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The Relationship Between Perfect and Imperfect Information in a Two-Action Risk-Sensitive Problem

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The ability to value information is a central feature of decision analysis and one of its most interesting areas of application. Unfortunately, general assertions regarding the drivers of information value or its properties have been difficult to formulate or have been disproved by counterexample. In this paper, we investigate the value of imperfect information relative to perfect information (RVOI). Within the context of a two-action decision problem with normal priors and exponential utility, we derive a closed-form solution for the value of information and demonstrate that the RVOI is maximal when the decision maker is indifferent between the two alternatives. In addition, we determine when the value of an information system providing a normally distributed signal with correlation coefficient ρ is equal to $\rho \times 100\%$ or $\rho^2 \times 100\%$ of the value of perfect information. These results deepen our understanding of information value and enable practitioners to estimate the value of imperfect information in particular settings.

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1. Introduction

The ability to value information is a central feature of decision analysis and one of its most interesting areas of application. Unfortunately, general statements regarding the drivers of information value or its properties have been difficult to formulate or have been disproved by counterexample. For example, Hilton (1981) proved that there is no general monotonic relationship between the degree of action flexibility and information value. That is, adding (removing) actions to (from) the decision-maker's feasible set does not necessarily increase (decrease) information value. The same paper also demonstrated that there is no general monotonic relationship between the degree of risk aversion and information value. Gould (1974) showed that increasing uncertainty in the prior distribution does not necessarily lead to larger valuations of information. Samson et al. (1989) demonstrated that value of information (VOI) is not generally additive across independent sources of uncertainty. For example, the VOI on random variables $\tilde{x}$ and $\tilde{y}$ together may be greater than (superadditive), less than (subadditive), or equal to the VOI on $\tilde{x}$ plus the VOI on $\tilde{y}$.

Despite these negative findings, progress has been made by studying stylized or canonical VOI formulations such as the two-action linear-loss (TALL) problem, which was analyzed extensively by Schlaifer, Raiffa, and Pratt (Schlaifer 1959, Raiffa and Schlaifer 1961, Pratt et al. 1995). In this problem, a risk-neutral decision maker (DM) chooses between two acts or alternatives (sometimes referred to as the go/no-go, accept/reject, act/wait, new/status-quo alternatives) with uncertain outcomes, which are a linear function of a random variable $\tilde{x}$.

Although general conclusions are unobtainable, the simple TALL structure facilitates limited conclusions regarding VOI drivers. For example, Mehrez and Stulman (1982) demonstrated that when considering only translations of the underlying probability distribution functions, the expected value of perfect information is maximized when the expected value of the accept alternative is equal to the value of the reject alternative (i.e., the DM is indifferent). Fatti et al. (1987) extended this result to the case of imperfect information, again, only under translations of the probability density functions. Delquié (2008) proved that the difference between the expected utility with information and the expected utility without information is maximal when the DM is indifferent under a weaker restriction regarding the change in the underlying distributions. Delquié extended this result to the VOI as long as the DM's utility function exhibits constant absolute risk aversion. In the case of risk aversion, Mehrez (1985) proved that when the expected value of the accept alternative is less than or equal to the value of the reject alternative, a risk-averse DM will never pay more for perfect information than will a risk-neutral DM.

When the random variable in the TALL problem is assumed to be normally distributed, a variant called the TALL problem with normal priors (TALL-N) is obtained. This further restriction enabled Schlaifer (1959) to derive closed-form solutions for the value of both perfect and imperfect information. This allowed Keisler to study the effect of changes in prior mean and standard deviation on information value (Keisler 2004) and the conditions underlying sub- and super-additivity in information value (Keisler 2005).

In this paper, we analyze the relationship between imperfect and perfect information in the two-action problem for risk-neutral DMs (i.e., the TALL problem) and risk-sensitive DMs whose utility function is exponential. We refer to the latter as the two-action exponential-utility (TAXU) problem and to the further restriction of normality as TAXU-N. In particular, we are interested in the ratio of the value of imperfect information to the value of perfect information (RVOI), or the relative value of information (relative to the gold standard of perfect information). For example, suppose a DM can purchase an information system[1] whose signal θ is correlated with the uncertain value of the accept alternative, with correlation coefficient ρ. How much is such a system worth relative to perfect information? An initial guess might be that it should be worth $\rho \times 100\%$ of perfect information because it is in some sense (but not always in the right sense, as we shall see) $\rho \times 100\%$ as "good" as perfect information. A slightly more sophisticated guess is that such a system should be worth $\rho^2 \times 100\%$ of perfect information, since ρ^2 is a measure of the proportional amount by which knowledge of θ reduces the variance of $\tilde{x}$ if the regression of $\tilde{x}$ on $\tilde{\theta}$ is linear, as is the case if $\tilde{x}$ and $\tilde{\theta}$ are jointly normally distributed, for example. Not surprisingly, neither of these answers is correct in all cases. They do obtain, however, in certain circumstances. For example, Pratt et al. (1995) demonstrated that $\text{RVOI} = \rho^2$ under quadratic loss and a linear regression function, and Fatti et al. (1987) proved that $\text{RVOI} = \rho$ within the context of the TALL-N problem when the DM is indifferent.

The contribution of this paper is twofold. First, we obtain a closed-form solution for the VOI in the TAXU-N problem, the TALL-N solution discussed above being a special case. Second, we provide insight into the drivers of information value in general and the RVOI in particular, in both the TALL-N and TAXU-N problems. For example, we demonstrate under what conditions this ratio would be equal to ρ or ρ^2. The result of Fatti et al. (1987) is shown to be a special case of our more general formulation.

The RVOI is of interest for several reasons. First, the value of perfect information is often easier to calculate than the value of imperfect information because the likelihood assessments and Bayesian calculations are trivial. This is not simply a matter of computation; Bayesian analysis often requires difficult assessments, which greatly limits its use in practice. For example, an oil company may be able to readily calculate the value of perfect information regarding reservoir property x. Yet, valuing a test that provides imperfect information regarding x may require difficult probabilistic assessments and modeling. Understanding the behavior of the RVOI, albeit in a simple setting, may provide a rough estimate of the test's potential value. Second, both theoretic and pedagogic settings benefit from understanding the behavior of imperfect information relative to perfect information. For example, as the prior uncertainty is increased, does imperfect information become more or less valuable relative to perfect information?

The strictness of the TAXU and TAXU-N assumptions results in a highly stylized problem, but this constraining of the problem space is necessary if we hope to derive general analytic results. The question is then for how many problems does the model we

[1] An information system is an information-gathering opportunity, such as research or testing, which provides an observable signal that is probabilistically related to the uncertainty in which we are interested but cannot directly observe (Demski 1972).

study capture their essential features and how well. We believe that both the TALL-N and TAXU-N models perform quite well in this regard. Many important decision opportunities are two-action problems, e.g., whether or not to launch a new product, invest or not in a risky project, or accept or reject a particular part in a quality-control setting. In addition, many random variables can be appropriately modeled as being normally distributed, perhaps after transformation. Finally, risk neutrality is a common assumption used in practice. When risk sensitivity is important, the exponential utility function is a robust approximation (Kirkwood 2004).

The remainder of this paper is organized as follows. In §2 we present a motivating example. In §3 we review the TALL-N problem. In §4 we introduce the TAXU-N problem, derive the closed-form solution for the VOI in this setting, and show that it converges to the TALL-N problem as the DM's utility function approaches risk neutrality. In §5 we analyze the RVOI in both the TALL-N and TAXU-N problems. Finally, we conclude in §6.

2. Motivating Example

In this section, we introduce an example that motivates our study of the value of information ratio and facilitates the understanding of subsequent results.

Suppose an oil company with a risk tolerance of \$100 million is considering drilling a well in an undeveloped area. Based on core samples and well logs, the company estimates that the net present value of the well is normally distributed with a mean of \$10 million and a standard deviation of \$20 million. Evaluating the standard normal cumulative at a value of 0.5 (\$10/\$20), the company finds that the probability of the well being a good investment (net present value >\$0) is 69%—implying a 31% chance of losing money. If the company does not drill, it will earn a sure \$0.

In an effort to improve its decision, the company is considering the acquisition of a seismic survey costing \$2 million. Modeling the value of seismic surveys is complicated and time consuming (Bickel et al. 2006, Pickering and Bickel 2006, Gibson et al. 2007). However, based on previous results in similar areas, the oil company's geophysicist believes the seismic results are correlated with the true value of the well with a correlation coefficient of 0.5.

Should the company acquire the survey? Should the company spend the time to model this information-gathering opportunity in detail? Is it possible to develop a "back of the envelope" value for the survey? How sensitive is this value to the company's estimate of the correlation coefficient (or other model parameters)? Would a better-quality seismic survey, with a 25% higher correlation coefficient, be worth 25% more?

As we shall soon see, the expected value of perfect information (EVPI) regarding the value of the well is almost \$4 million. Based on this value, the company considers two simple arguments: (1) The EVPI is substantially higher than the cost of the (imperfect) survey and therefore the survey must be worthwhile.[2] (2) The survey is 50% as good as perfect information because its correlation coefficient is 0.5. Therefore, the survey is worth \$2 million—equal to its cost.[3]

Experts in decision analysis will recognize that neither of these arguments is correct. Do we therefore recommend that the company explicitly model the information-gathering opportunity? Or, can we provide a better first-cut estimate of the survey's value?

3. The Two-Action Linear-Loss Problem

In this section, we review the version of the two-action linear-loss problem that we study in this paper. This serves to introduce notation and build intuition regarding the exponential-utility solution, which is more complicated.

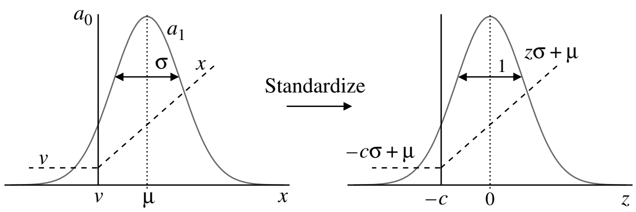
Suppose a risk-neutral DM faces a choice between two alternatives: Alternative 0 (a_0) and Alternative 1 (a_1). Alternative 0 yields a certain payoff of v, while a_1 produces an uncertain payoff characterized by a random variable $\tilde{x}$ with mean $\mu = E[\tilde{x}]$, where E is the expectation operator.[4] In the absence of additional information, the DM's optimal choice is a_0 if

[2] Some practitioners use this argument to gather information. For example, see Portella et al. (2003).

[3] In the author's experience, practitioners and students find this argument compelling.

[4] We could instead assume that a_0 produces a random payoff $\tilde{v}$, independent of $\tilde{x}$, but such an assumption would not materially change our results and complicates the notation somewhat.

Figure 1 TALL-N Problem When the Decision Maker Initially Prefers the Risky Alternative



$\mu < v$ and a_1 if $\mu \geq v$; the DM is indifferent if $\mu = v$. The DM's expected value (or certain equivalent, since he/she is risk neutral) without additional information is $\max(v, \mu)$.

3.1. The TALL Problem with Normal Priors

We now assume that $\tilde{x}$ is normally distributed with mean μ and standard deviation σ, obtaining the TALL problem with normal priors (TALL-N). The left-hand graphic in Figure 1 displays a case where the decision maker initially prefers the risky alternative.

Expected Value of Perfect Information. The dotted lines in Figure 1 show the DM's payoff function with perfect information. If x is less than v, then the decision maker will choose a_0 and earn v. On the other hand, if x is greater than or equal to v, the DM will choose a_1 and earn x. The expected value with perfect information is equal to $E[\max(v, x)]$ and may be readily calculated. We first standardize $\tilde{x}$ by letting $z = (x - \mu)/\sigma$ or $x = z\sigma + \mu$. This yields the right-hand diagram in Figure 1, where $c = (\mu - v)/\sigma$. c is a measure of the distance or divergence between the alternatives measured in units of standard deviation, which we will refer to as the "coefficient of divergence." The expected value *with* perfect information is then

$$E[\max(v, x)] = \mu - c\sigma\Phi(-c) + \sigma\int_{-c}^{\infty} z\phi(z)\,dz$$
$$= \mu - c\sigma\Phi(-c) + \sigma\phi(c), \qquad (1)$$

where Φ is the standard normal cumulative probability function and ϕ is the standard normal probability density function. The integral in Equation (1) is the mean of $\tilde{z}$ given that z is greater than $-c$, which is simply $\phi(-c) - \phi(\infty) = \phi(-c) = \phi(c)$ since $\tilde{z}$ is standard normal. The expected value *of* perfect information (EVPI) is $E[\max(v, x)] - \max(v, \mu)$ or

$$EVPI = \begin{cases} \sigma[\phi(c) - c\Phi(-c)] & \mu \geq v \\ \sigma[\phi(c) + c\Phi(c)] & \mu < v. \end{cases} \qquad (2)$$

The term in brackets is known as the unit-normal linear-loss integral (Schlaifer 1959, p. 453).

Motivating Example: Returning to the example introduced in §2, with $c = \$10/\$20 = 0.5$ and $\sigma = \$20$ million, we find that $EVPI$ (in millions) $= \$20[\phi(0.5) - 0.5\Phi(-0.5)] = \$3.96$.

Expected Value of Imperfect Information. Now suppose the DM can obtain a free information system (IS) Θ_ρ whose signal θ is jointly normally distributed with $\tilde{x}$ with positive correlation coefficient ρ.[5] We will refer to such an information system as a "ρ-information system" or ρ-IS. The DM's expected value *with* the free IS is $E[\max(v, E[\tilde{x} \mid \theta])]$. The expected value *of* information (EVI) is

$$EVI_\Theta(\tilde{x}) = E[\max(v, E[\tilde{x} \mid \theta])] - \max(v, \mu). \qquad (3)$$

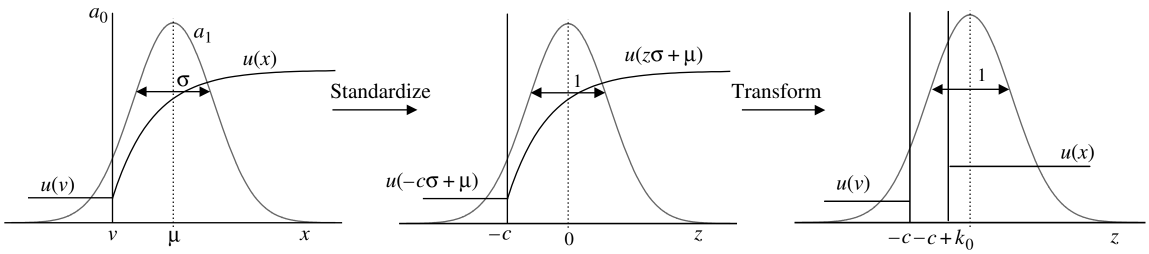
Without loss of generality, we assume that the information signal is unit normal and therefore the conditional mean of $\tilde{x}$ given θ is $E[\tilde{x} \mid \theta] = \mu + \rho\sigma\theta$, which will exceed v as long as $\theta > (v - \mu)/(\rho\sigma) = -\rho^{-1}c$. The expected value of information is then (see Appendix A1 for derivation)

$$EVI_\rho = \begin{cases} \rho\sigma[\phi(\rho^{-1}c) - \rho^{-1}c\Phi(-\rho^{-1}c)] & \mu \geq v \\ \rho\sigma[\phi(\rho^{-1}c) + \rho^{-1}c\Phi(\rho^{-1}c)] & \mu < v. \end{cases} \qquad (4)$$

A 1-IS is perfect information or clairvoyance and has a value of $EVI_1 \equiv EVPI$. Notice that since $\rho^{-1}c =$

[5] If we had allowed for random v then we would have required that $\tilde{v}$ be independent of $\tilde{x}$ and θ.

Figure 2 Expected Utility of Perfect Information in TAXU-N Problem



$(\mu - v)/(\rho\sigma)$, the expected value of imperfect information is equivalent to expected value of perfect information with a standard deviation equal to $\rho\sigma$.[6]

4. The Two-Action Exponential-Utility Problem

We now extend the TALL problem by allowing the DM to be risk sensitive. Specifically, we will assume that the DM's utility function is exponential and exhibits constant absolute risk aversion (CARA). We refer to this as the two-action exponential-utility (TAXU) problem. In the remainder of this paper, we assume that the DM's utility function is monotonically increasing such that $u(x) = -\text{sgn}(R)\text{Exp}(-x/R)$, where R is the DM's risk tolerance. The monotonically decreasing case is similar and is omitted.

The DM is now concerned with the certain equivalent $\hat{x}$ of the risky alternative. In the absence of additional information, the DM's optimal choice is a_0 if $\hat{x} < v$ and a_1 if $\hat{x} \geq v$. The DM's expected utility without additional information is $\max(u(v), Eu(\tilde{x})) = \max(u(v), u(\hat{x}))$, where $Eu(\tilde{x}) \equiv E[u(\tilde{x})]$. The DM's certain equivalent without additional information is $u^{-1}(\max(u(v), Eu(\tilde{x}))) = \max(v, \hat{x})$.

As before, suppose the DM can obtain a free information system that yields a signal θ regarding the value of $\tilde{x}$. The DM's expected utility and certain equivalent of the free IS are $E[\max(u(v), Eu(\tilde{x} \mid \theta))]$ and $u^{-1}(E[\max(u(v), Eu(\tilde{x} \mid \theta))])$, respectively. We define the VOI on random variable $\tilde{x}$ under information system Θ as the DM's indifferent buying price, which under CARA is

$$VOI_\Theta(\tilde{x}) = u^{-1}(E[\max(u(v), Eu(\tilde{x} \mid \theta))]) - \max(v, \hat{x}). \quad (5)$$

4.1. The TAXU Problem with Normal Priors

Because general conclusions regarding VOI are difficult to draw, we maintain a simple structure by assuming that $\tilde{x}$ is normally distributed with mean μ and finite standard deviation σ. We refer to this as the TAXU problem with normal priors (TAXU-N). The normality assumption in conjunction with the assumption of exponential utility yields a certain equivalent for $\tilde{x}$ of $\hat{x} = \mu - \sigma^2/(2R)$ (Howard 1971). The term $\sigma^2/(2R)$ is the risk premium.

Value of Perfect Information. The DM's expected utility with perfect information is (see Appendix A2 for derivation)

$$E[u(\max(v, x))] = u(v)\Phi(-c) + u(\hat{x})[1 - \Phi(-c + k)], \quad (6)$$

where $k = \sigma/R$. We refer to k as the "coefficient of relative risk" because it measures the degree of uncertainty in $\tilde{x}$, in units of risk tolerance; larger values of k imply larger risk premiums or, speaking loosely, greater discounting of the expected value because uncertainty is greater relative to the DM's risk tolerance. The condition $\hat{x} \geq v$ is equivalent to the condition $2c \geq k$.

Equation (6) can be readily understood in the context of Figure 2. We begin by standardizing $\tilde{x}$ and then transforming the expected-utility calculation to the equivalent case given in Equation (6) and depicted in the right-hand graphic of Figure 2. The first part of Equation (6) is straightforward: the DM will earn $u(v)$

[6] The TALL problem can be extended to the case of a lognormal prior. See Appendix A7 for the EVI formula.

with probability $\Phi(-c)$. The second part is more complicated: the DM will earn the utility of a conditional certain equivalent with probability $1-\Phi(-c)$. This case is equivalent to earning the utility of the gamble $u(\hat{x})$ with a probability equal to $1-\Phi(-c+k)$, where the size of the adjustment k depends upon the uncertainty of the gamble and the DM's risk tolerance.

The value of perfect information (VOI_1) is (see Appendix A3 for derivation)

$$VOI_1 = \begin{cases} -R\ln[e^{ck-k^2/2}\Phi(-c)+\Phi(c-k)] \\ \quad 2c \geq k \ (\hat{x} \geq v) \\ -R\ln[\Phi(-c)+e^{-ck+k^2/2}\Phi(c-k)] \\ \quad 2c < k \ (\hat{x} < v). \end{cases} \tag{7}$$

Equation (7) is the analogue to Equation (2), which yields the expected value (risk neutral) of perfect information.

Motivating Example: Returning to our motivating example with $c = 0.5$, $k = 0.2$, and $R = \$100$ million, we find that $VOI_1 = \$4.90$ million. Thus, the company's risk aversion has resulted in an increase in the value of perfect information.

Value of Imperfect Information. To analyze the value of imperfect information, we maintain our assumption that the information signal and the value of the risky alternative are jointly normal with positive correlation coefficient ρ.

Given a particular signal θ, the DM's posterior certain equivalent is $\hat{x}_\theta = \mu + \rho\sigma\theta - \sigma^2(1-\rho^2)/(2R)$. Let VWI_ρ be the DM's certain equivalent with a costless ρ-IS or the Value with Information (VWI). The DM's expected utility *with* the information system is then $u(VWI_\rho) = E[u(\max(v, \hat{x}_\theta))]$, which is equal to (see Appendix A4 for derivation)

$$u(VWI_\rho) = u(v)\Phi\left(\frac{v - E[\hat{x}_\theta]}{\rho\sigma}\right) + u(\hat{x})\left[1 - \Phi\left(\frac{v - E[\hat{x}_\theta]}{\rho\sigma} + \rho k\right)\right]. \tag{8}$$

The form of Equation (8) is similar to that of Equation (6), with the following differences. Instead of the difference $v-\mu$, we are now concerned with the difference between v and $E[\hat{x}_\theta] = \mu - \sigma^2(1-\rho^2)/(2R)$, which is the expected posterior certain equivalent.[7] In addition, as was true in the risk-neutral case, the relevant standard deviation is $\rho\sigma$ rather than simply σ.

We may write Equation (8) more compactly as

$$u(VWI_\rho) = u(v)\Phi\left(-\frac{c}{\rho} + \frac{k}{\rho}\frac{1-\rho^2}{2}\right) + u(\hat{x})\Phi\left(\frac{c}{\rho} - \frac{k}{\rho}\frac{1+\rho^2}{2}\right). \tag{9}$$

Because the DM's utility function exhibits constant risk aversion, his/her indifferent buying price for the costless ρ-IS, or the VOI, is (see Appendix A5 for derivation)

$$VOI_\rho = -R\ln[-\text{sgn}(R)u(VWI_\rho)] - \max(v, \hat{x})$$
$$= \begin{cases} -R\ln\left[e^{ck-k^2/2}\Phi\left(-\frac{c}{\rho}+\frac{k}{\rho}\frac{1-\rho^2}{2}\right) + \Phi\left(\frac{c}{\rho}-\frac{k}{\rho}\frac{1+\rho^2}{2}\right)\right] & 2c \geq k \ (\hat{x} \geq v) \\ -R\ln\left[\Phi\left(-\frac{c}{\rho}+\frac{k}{\rho}\frac{1-\rho^2}{2}\right) + e^{-ck+k^2/2}\Phi\left(\frac{c}{\rho}-\frac{k}{\rho}\frac{1+\rho^2}{2}\right)\right] & 2c < k \ (\hat{x} < v). \end{cases} \tag{10}$$

Note that the term $\text{sgn}(R)$ cancels in the certain equivalent formula. The value of perfect information is found by setting $\rho = 1$ in Equation (10), to obtain Equation (7). Thus, we have obtained a closed-form solution for the value of information in the TAXU-N problem. We believe Equations (7) and (10) are new, as previous authors (Schlaifer 1959, Raiffa and Schlaifer 1961, Winkler 1972, Keisler 2004) have focused on the TALL-N problem.

We can obtain the TALL-N solution via the following procedure. First, find the utility of the value

[7] Of course, the expected posterior certain equivalent is simply μ in the risk-neural case.

of information (the indifferent buying price). From Equation (10), this is

$$u(VOI_\rho) = \begin{cases} -\text{sgn}(R)\left[e^{ck-k^2/2}\Phi\left(-\frac{c}{\rho}+\frac{k}{\rho}\frac{1-\rho^2}{2}\right) + \Phi\left(\frac{c}{\rho}-\frac{k}{\rho}\frac{1+\rho^2}{2}\right)\right] & 2c\geq k\ (\hat{x}\geq v) \\ -\text{sgn}(R)\left[\Phi\left(-\frac{c}{\rho}+\frac{k}{\rho}\frac{1-\rho^2}{2}\right) + e^{-ck+k^2/2}\Phi\left(\frac{c}{\rho}-\frac{k}{\rho}\frac{1+\rho^2}{2}\right)\right] & 2c< k\ (\hat{x}< v). \end{cases} \quad (11)$$

Next, add 1 to the utilities in Equation (11) and divide by $1 - e^{-\gamma}$, where $\gamma = R^{-1}$. This places the original utility function in the form

$$u(y) = \frac{1-\text{sgn}(\gamma)\text{Exp}[-\gamma y]}{1-\text{Exp}[-\gamma]},$$

which shows that linear utility is a limiting case of exponential. Next, apply l'Hôpital's rule and take the limit as γ approaches zero (see Appendix A6 for derivation). This yields the expected value of information (EVI) as given in Equation (4). Letting $\rho = 1$ in Equation (4) yields the expected value of perfect information given in Equation (2). Thus, the TALL-N problem is a limiting case of the TAXU-N problem as R approaches infinity.

5. The Ratio of Imperfect to Perfect Information Value

5.1. TALL-N Problem

If we set $\mu = v$ in Equation (4), and thereby $c = 0$, then $EVI_\rho = \rho\sigma\phi(0) = \rho EVI_1$. In other words, in the TALL-N problem the expected value of imperfect information is simply ρ times the value of perfect information when the DM is indifferent, as demonstrated by Fatti et al. (1987). In general, the ratio of the value of imperfect to perfect information in the TALL-N problem is

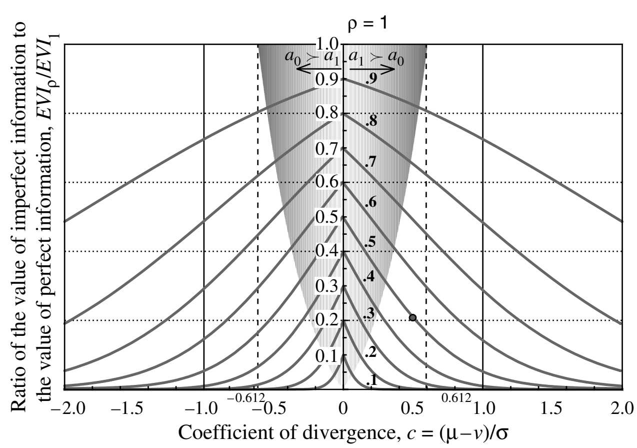
$$\frac{EVI_\rho}{EVI_1} = \begin{cases} \rho & \mu = v \\ \dfrac{\rho\phi(\rho^{-1}c)-c\Phi(-\rho^{-1}c)}{\phi(c)-c\Phi(-c)} & \mu > v \\ \dfrac{\rho\phi(\rho^{-1}c)+c\Phi(\rho^{-1}c)}{\phi(c)+c\Phi(c)} & \mu < v. \end{cases} \quad (12)$$

Therefore, EVI_ρ/EVI_1 is *a function only of the correlation coefficient and the coefficient of divergence*, c. Figure 3 displays Equation (12) for correlation coefficients ranging from 0.1 to 1.0. The DM is indifferent between a_0 and a_1 when $c = 0$. When $c > 0$ he/she prefers a_1 and when $c < 0$ he/she prefers a_0. We see that EVI_ρ/EVI_1 is symmetric about $c = 0$ and falls quickly as the absolute value of c is increased. The highlighted area is the region where the ratio is at least ρ^2. This region was found numerically because Equation (12) cannot be solved explicitly for c.

Figure 3 confirms that moving the DM away from the point of indifference by either increasing the absolute difference between the two alternatives or decreasing the standard deviation of the risky alternative (assuming $\mu \neq v$) lowers the value of imperfect information relative to perfect information.

Motivating Example: Returning to the example presented in §2, we note that $c = 0.5$, $\rho = 0.5$, and therefore $EVI_{0.5}/EVI_1 = 0.21$. Thus, if the company were risk neutral, the survey would only be worth about

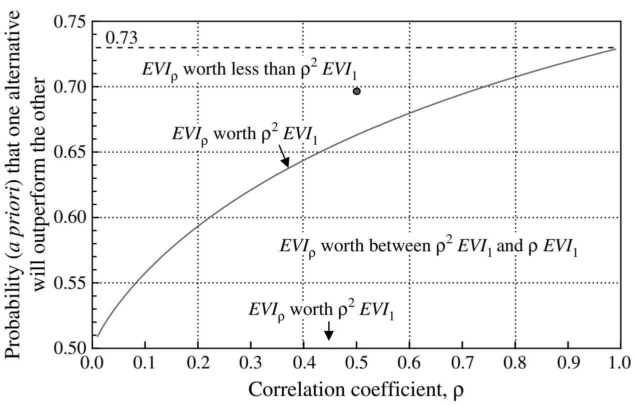
Figure 3 **The Ratio of the Expected Value of Imperfect Information to the Expected Value of Perfect Information as a Function of the Coefficient of Divergence in the TALL-N Problem**



\$0.84 million (21% of EVI_1), which is less than half its cost. The small dot in Figure 3 represents the value of the survey; this point is quite distant from the location where the survey would be value-adding (50% of EVI_1). In fact, based on Figure 3, or by numerically solving Equation (12) for ρ, the \$2 million survey would require a correlation coefficient above 0.7 to be value-adding. Similarly, we could use Figure 3 to explore the implications of other changes in model parameters. For example, if the well's standard deviation were only \$10 million, then $c = 1$ and imperfect information would be worth only about 5% of perfect information.

The size of the highlighted region where imperfect information is worth at least $\rho^2 \times 100\%$ of perfect information decreases with the correlation coefficient. Not only is EVI_ρ always less than ρEVI_1 for nonzero c, but it is often less than $\rho^2 EVI_1$. In fact, if the absolute value of c is greater than 0.612 (i.e., there is at least a 73% chance that one alternative will outperform the other), then EVI_ρ/EVI_1 is less than ρ^2 for any level of correlation. Figure 4 presents this relationship as a function of the correlation coefficient and the probability that one alternative will produce a better outcome than the other. For example, if there is a 69% chance (a priori) that one alternative will outperform the other, as in our example, then information systems with correlations less than 0.67 are worth less than $(0.67)^2 \times 100\%$ of perfect information. This case is identified by the small dot in Figure 4.

Figure 4 **Sensitivity of TAXU VOI to Prior Uncertainty and Correlation Coefficient**



5.2. TAXU-N Problem

The RVOI in the TAXU-N problem is

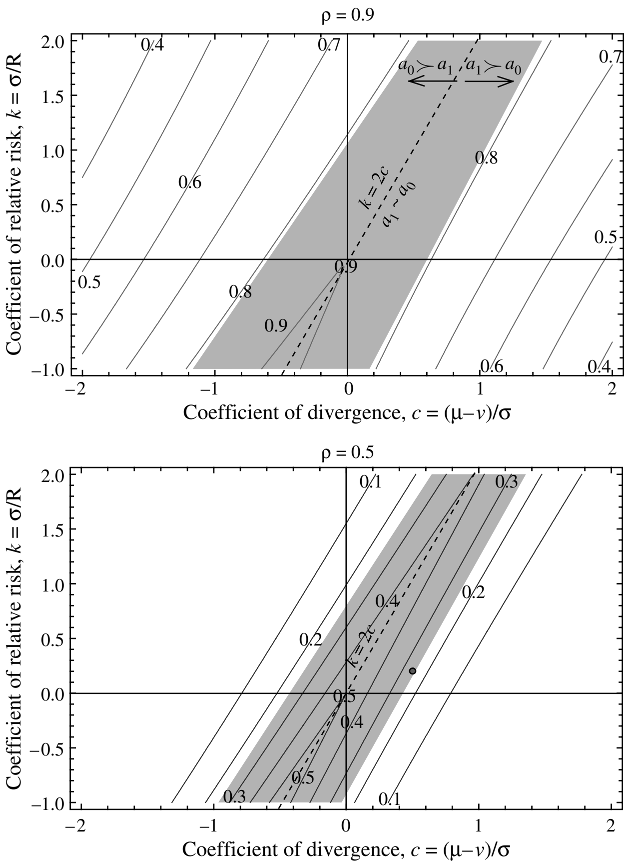
$$RVOI_\rho = \frac{VOI_\rho}{VOI_1} = \begin{cases} \dfrac{\ln\left[e^{ck-k^2/2}\Phi\left(-\frac{c}{\rho}+\frac{k}{\rho}\frac{1-\rho^2}{2}\right)+\Phi\left(\frac{c}{\rho}-\frac{k}{\rho}\frac{1+\rho^2}{2}\right)\right]}{\ln[e^{ck-k^2/2}\Phi(-c)+\Phi(c-k)]} & 2c \geq k\ (\hat{x} \geq v) \\[2ex] \dfrac{\ln\left[\Phi\left(-\frac{c}{\rho}+\frac{k}{\rho}\frac{1-\rho^2}{2}\right)+e^{-ck+k^2/2}\Phi\left(\frac{c}{\rho}-\frac{k}{\rho}\frac{1+\rho^2}{2}\right)\right]}{\ln[\Phi(-c)+e^{-ck+k^2/2}\Phi(c-k)]} & 2c < k\ (\hat{x} < v), \end{cases} \tag{13}$$

which is *a function only of the correlation coefficient and the two fundamental ratios* c *and* k. The value of imperfect information relative to the value of perfect information in the TAXU-N problem is completely determined by the accuracy of the information system (ρ), the divergence between the two alternatives measured in units of standard deviation (c), and the ratio of the standard deviation to the DM's risk tolerance (k). The specific mean values of the alternatives, their risk, or the DM's risk tolerance are not determining factors.

Figure 5 plots constant $RVOI_\rho$ contours as a function of c and k for correlations of 0.5 and 0.9. The axis where $k = 0$ is equivalent to the TALL-N results presented in Figure 3. The DM is indifferent along the line where $k = 2c$. We see that when $k > 0$ (i.e., a risk-averse DM), imperfect information is worth less than ρVOI_1; a risk-averse DM will value a ρ-IS at less than $\rho \times 100\%$ of clairvoyance. However, a risk-preferring DM ($k < 0$) may value a ρ-IS *more* highly than ρVOI_1, as depicted by the region enclosed by the contours equal to ρ in the lower left quadrant of each graph. The highlighted region is the set of points (c, k) such that imperfect information is worth at least $\rho^2 VOI_1$ (this region was again found numerically). As with the TALL-N problem, the size of this region decreases with decreasing information-system accuracy.

The effect of changing risk attitude depends on which side of the indifference line the DM starts. For example, if the DM initially prefers a_1 to a_0,

Figure 5 **RVOI as a Function of the Coefficient of Divergence and Coefficient of Relative Risk in the TAXU-N Problem**



then increasing risk aversion (decreasing R, increasing k) will increase RVOI until the point of indifference, beyond which further increases in risk aversion will reduce RVOI. A similar argument can be made regarding changes in the mean difference between the two alternatives. Changes in the standard deviation are more complex and encompass several cases:

Case 1: If both c and k are positive, then increasing uncertainty will increase the RVOI if the DM initially prefers the risky alternative and decrease it otherwise.

Case 2: If both c and k are negative, then increasing uncertainty will decrease the RVOI if the DM initially prefers the risky alternative and increase it otherwise.

Case 3: If c and k are of opposite sign, then the effect of increasing uncertainty is indeterminate. For example, if $\rho = 0.9$, $\sigma = 10$, $c = -1$, and $k = 0.1$, then the RVOI is 0.71 and increasing σ to 15 increases the RVOI to 0.77. If instead $\rho = 0.9$, $\sigma = 10$, $c = -0.1$, and $k = 1$, then the RVOI is 0.80 and increasing σ to 15 decreases the RVOI to 0.76.

Despite the complexities, these effects can be summarized as follows: moving the DM away from the line of indifference ($k = 2c$) decreases the RVOI, whereas moving the DM closer to the line of indifference increases the RVOI.

Motivating Example: Returning to our example, we have $c = 0.5$, $k = 0.2$, and $VOIR_{0.5} = 0.26$. This value is depicted by the small dot in the lower panel of Figure 5. Taking the company's risk aversion into account increases the value of the survey from \$0.84 million (21% of EVI_1) to \$1.26 million (26% of VOI_1), which is still less than its cost. Furthermore, the company could use Figure 5 (or similar graphs) to understand how changes in its estimates affect information value. This sensitivity analysis is possible because the RVOI is a dimensionless quantity and does not depend explicitly on the absolute model parameters, but rather on their ratios as defined by c and k. Such a general graph is not possible for VOI because this quantity depends upon the specific model parameters.

Given a particular c, the RVOI will be maximized when $k = 2c$; imperfect information is worth the most relative to perfect information when the DM is indifferent. Along the line $k = 2c$,

$$RVOI_\rho^* = \left(\frac{VOI_\rho}{VOI_1}\right)^* = \frac{\ln[2\Phi(-\rho c)]}{\ln[2\Phi(-c)]}, \tag{14}$$

where the * superscript denotes that this is the maximum RVOI for a particular c.[8] This ratio is plotted in Figure 6 and involves three intuitive limits:

As $c \to 0, RVOI_\rho^* \to \rho$: This was shown in the TALL-N problem, where $RVOI_\rho = \rho$ if both c and k are 0. We cannot simply set c equal to 0 in Equation (14), implying that k is also 0, because our formulation assumed that the DM's risk tolerance was finite ($k \neq 0$).

As $c \to -\infty$, $RVOI_\rho^* \to 1$: Negative values of c imply negative risk tolerances along the line $k = 2c$. Therefore, a risk-preferring DM could value imperfect information almost as highly as perfect information if the status-quo alternative is significantly better than the risky alternative. For example, if $c = -4$ (which

[8] The maximum value of imperfect information is equal to $VOI_\rho^* = -R\ln[2\Phi(-\rho c)]$.

implies $k = -8$), a risk-preferring DM would value a 0.8-IS at 99.9% of clairvoyance and a 0.6-IS at 98.8% of clairvoyance. This extreme result obtains because any information is almost worthless in this case; the probability that the risky alternative is better than the certain alternative is $\Phi(-4) \approx 0$.

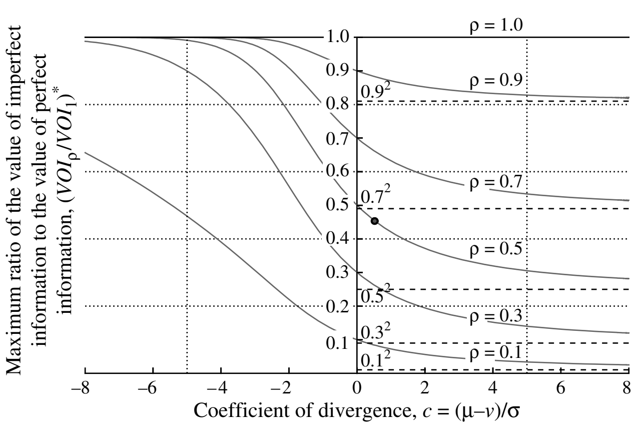
As $c \to \infty$, $RVOI_\rho^* \to \rho^2$: Positive values of c imply positive risk tolerances along the line $k = 2c$. As c approaches infinity, the limit of Equation (14) is indeterminate in its current form. Applying l'Hôpital's rule, we obtain the new limit

$$\lim_{c\to\infty} \rho \frac{\lambda(\rho c)}{\lambda(c)}, \tag{15}$$

where λ is the normal hazard function. $\lambda(c)$ approaches c as c approaches infinity. Therefore, both the limit in Equation (15) and $RVOI_\rho^*$ approach ρ^2. As discussed earlier, ρ^2 is the proportional amount by which knowledge of θ reduces the variance of $\tilde{x}$ when the regression of $\tilde{x}$ on $\tilde{\theta}$ is linear—as it is in TAXU-N. Thus, within the context of the TAXU-N problem, a risk-averse DM could value a ρ-IS based on its ability to reduce the variance of $\tilde{x}$.

Motivating Example: The maximum RVOI in our drilling example occurs when $k = 2c = 1$, instead of $k = 0.2$. This value of k could be achieved by increasing the company's risk aversion (i.e., decreasing its risk tolerance) or increasing the well's uncertainty. At this value of k (with $c = 0.5$ and $\rho = 0.5$), which is represented by the small dot in Figure 6, the seismic survey is worth approximately 46% of perfect information or \$1.86 million—again less than its cost of \$2 million. Since this is the most that any survey could be worth with $c = 0.5$ and $\rho = 0.5$, the company's risk attitude is immaterial and the survey should not be purchased.

Figure 6 **The Maximum RVOI as a Function of the Coefficient of Divergence in the TAXU Problem**



6. Conclusion

Despite the absence of simple rules of thumb regarding the value of imperfect information relative to perfect information, we can provide decision makers with some guidance. For example, an information system with correlation coefficient ρ will never be valued by non-risk-preferring decision makers at more than $\rho \times 100\%$ of the value of perfect information. In fact, an upper limit of $\rho^2 \times 100\%$ does not seem to be unreasonable. This finding helps decision makers understand the importance of information-system accuracy. For example, changing the accuracy of an information system by a factor of β will change its value by a factor of β^2.

If the DM wants to use a rule of thumb for valuing imperfect information as a fraction of perfect information, then he/she needs to understand that this fraction varies greatly depending on the distance between the alternatives. Figures 4 through 6 provide the appropriate fraction and offer the possibility of quickly estimating an information system's value so that a decision regarding more detailed modeling can be made. For example, if the chance that one alternative is better than the other is at least 73%, then the expected value of imperfect information is worth at most $\rho^2 \times 100\%$ of the expected value of perfect information.

While the model structure studied here is simple, it produces valuable and sometimes surprising results. For researchers, it deepens our understanding of information value in general and the relative value of imperfect information in particular. For practitioners, it provides guidance regarding the drivers of information value and facilitates rough VOI estimates. However, as evidenced by the mostly negative results concerning the properties of VOI discussed at the outset, great care must be taken in extrapolating these results to different settings.

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Appendix

A1. Expected Value of Imperfect Information

Let $a=(v-\mu)/(\rho\sigma)=-\rho^{-1}c$ and $b=\infty$. The expected value with imperfect information is

$$\begin{aligned}\int_{-\infty}^{a} vf(\theta)\,d\theta+\int_{a}^{\infty}(\mu+\rho\sigma\theta)f(\theta)\,d\theta\\ &=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{a} ve^{-(1/2)z^2}\,dz+\frac{1}{\sqrt{2\pi}}\int_{a}^{\infty}(\mu+\rho\sigma\theta)e^{-(1/2)\theta^2}\,d\theta\\ &=v\Phi(a)+\mu\frac{1}{\sqrt{2\pi}}\int_{a}^{\infty}e^{-(1/2)\theta^2}\,d\theta+\rho\sigma\frac{1}{\sqrt{2\pi}}\int_{a}^{\infty}\theta e^{-(1/2)\theta^2}\,d\theta\\ &=v\Phi(a)+\mu(1-\Phi(a))-\rho\sigma\frac{1}{\sqrt{2\pi}}e^{-(1/2)\theta^2}\Big|_{a}^{\infty}\\ &=v\Phi(a)+\mu(1-\Phi(a))+\rho\sigma\phi(a)\\ &=\mu+\rho\sigma\phi(a)+(v-\mu)\Phi(a)\\ &=\mu+\rho\sigma\phi(-\rho^{-1}c)-\sigma c\Phi(-\rho^{-1}c).\end{aligned}$$

The expected value of information is

$$EVI_\rho=\mu+\rho\sigma\phi(-\rho^{-1}c)-\sigma c\Phi(-\rho^{-1}c)-\max(v,\mu).$$

If $\mu\ge v$, then

$$\begin{aligned}EVI_\rho&=\rho\sigma\phi(-\rho^{-1}c)-\sigma c\Phi(-\rho^{-1}c)\\ &=\sigma[\rho\phi(-\rho^{-1}c)-c\Phi(-\rho^{-1}c)].\end{aligned}$$

If $\mu<v$, then

$$\begin{aligned}EVI_\rho&=\mu+\rho\sigma\phi(-\rho^{-1}c)-\sigma c\Phi(-\rho^{-1}c)-v\\ &=-(v-\mu)-\sigma c\Phi(-\rho^{-1}c)+\rho\sigma\phi(-\rho^{-1}c)\\ &=\sigma c-\sigma c\Phi(-\rho^{-1}c)+\rho\sigma\phi(-\rho^{-1}c)\\ &=\sigma c(1-\Phi(-\rho^{-1}c))+\rho\sigma\phi(-\rho^{-1}c)\\ &=\sigma c\Phi(\rho^{-1}c)+\rho\sigma\phi(\rho^{-1}c)\\ &=\rho\sigma[\phi(\rho^{-1}c)+\rho^{-1}c\Phi(\rho^{-1}c)].\end{aligned}$$

A2. Expected Utility with Perfect Information

Using the middle diagram in Figure 2, the expected utility with perfect information is

$$\begin{aligned}u(VwPI)&=u(-c\sigma+\mu)\Phi(-c)+\int_{-c}^{\infty}u(z\sigma+\mu)f(z)\,dz\\ &=e^{-\mu/R}\left[u(-c\sigma)\Phi(-c)-\operatorname{sgn}(R)\int_{-c}^{\infty}e^{-z\sigma/R}f(z)\,dz\right]\\ &=e^{-\mu/R}\bigg[u(-c\sigma)\Phi(-c)\\ &\qquad-\operatorname{sgn}(R)\frac{1}{\sqrt{2\pi}}\int_{-c}^{\infty}e^{-(1/2)z^2-(\sigma/R)z}\,dz\bigg].\end{aligned}$$

To evaluate the integral, note the following:

$$\begin{aligned}\frac{1}{\sqrt{2\pi}}\int_{x}^{\infty}e^{-(1/2)z^2-az}\,dz&=\frac{1}{\sqrt{2\pi}}\int_{x}^{\infty}e^{-(1/2)(z+a)^2+(1/2)a^2}\,dz\\ &=e^{(1/2)a^2}\frac{1}{\sqrt{2\pi}}\int_{x}^{\infty}e^{-(1/2)(z+a)^2}\,dz\\ &=e^{(1/2)a^2}\frac{1}{\sqrt{2\pi}}\int_{x+a}^{\infty}e^{-(1/2)z^2}\,dz\\ &=e^{(1/2)a^2}(1-\Phi(x+a))\\ &=e^{(1/2)a^2}\Phi(-x-a).\end{aligned}$$

Therefore,

$$\begin{aligned}u(VwPI)\\ &=e^{-\mu/R}\left[u(-c\sigma)\Phi(-c)-\operatorname{sgn}(R)\frac{1}{\sqrt{2\pi}}\int_{-c}^{\infty}e^{-(1/2)z^2-(\sigma/R)z}\,dz\right]\\ &=e^{-\mu/R}[u(-c\sigma)\Phi(-c)-\operatorname{sgn}(R)e^{(1/2)(\sigma/R)^2}\Phi(c-R^{-1}\sigma)]\\ &=u(-c\sigma+\mu)\Phi(-c)-\operatorname{sgn}(R)e^{(1/2)(\sigma/R)^2-(\mu/R)}\Phi(c-R^{-1}\sigma)\\ &=u(v)\Phi(-c)-\operatorname{sgn}(R)e^{-R^{-1}(\mu-\sigma^2/(2R))}\Phi(c-R^{-1}\sigma)\\ &=u(v)\Phi(-c)+u(\hat{x})[1-\Phi(-c+k)].\end{aligned}$$

A3. Value of Perfect Information

The value of perfect information is then

$$\begin{aligned}VOI_1\\ &=-R\ln[-u(v)\Phi(-c)-u(\hat{x})[1-\Phi(-c+k)]]-\max(v,\hat{x})\\ &=-R\ln[e^{-v/R}\Phi(-c)-e^{-\hat{x}/R}[1-\Phi(-c+k)]]-\max(v,\hat{x}).\end{aligned}$$

Note that the $\operatorname{sgn}(R)$ term cancels in the certain equivalent formula.

If $\hat{x}\ge v$, then

$$\begin{aligned}VOI_1&=-R\ln[e^{-v/R}\Phi(-c)+e^{-\hat{x}/R}(1-\Phi(-c+k))]-\hat{x}\\ &=-R\ln[e^{-v/R}\Phi(-c)+e^{-\hat{x}/R}(1-\Phi(-c+k))]-R\ln[e^{\hat{x}/R}]\\ &=-R\ln[e^{\hat{x}/R}e^{-v/R}\Phi(-c)+e^{\hat{x}/R}e^{-\hat{x}/R}(1-\Phi(-c+k))]\\ &=-R\ln[e^{(\hat{x}-v)/R}\Phi(-c)+(1-\Phi(-c+k))]\\ &=-R\ln[e^{ck-k^2/2}\Phi(-c)+\Phi(c-k)],\end{aligned}$$

since

$$\begin{aligned}e^{(\hat{x}-v)/R}&=e^{(\mu-\sigma^2/(2R)-v)/R}=e^{(\mu-v)/R-\sigma^2/(2R^2)}\\ &=e^{\sigma(\mu-v)/(R\sigma)-\sigma^2/(2R^2)}=e^{ck-k^2/2}.\end{aligned}$$

If $\hat{x}<v$, then

$$\begin{aligned}VOI_1&=-R\ln[e^{-v/R}\Phi(-c)+e^{-\hat{x}/R}(1-\Phi(-c+k))]-v\\ &=-R\ln[e^{-v/R}\Phi(-c)+e^{-\hat{x}/R}(1-\Phi(-c+k))]-R\ln[e^{v/R}]\\ &=-R\ln[e^{v/R}e^{-v/R}\Phi(-c)+e^{v/R}e^{-\hat{x}/R}(1-\Phi(-c+k))]\\ &=-R\ln[\Phi(-c)+e^{(v-\hat{x})/R}(1-\Phi(-c+k))]\\ &=-R\ln[\Phi(-c)+e^{-ck+k^2/2}\Phi(c-k)]\end{aligned}$$

A4. Expected Utility with Imperfect Information
Assume $\tilde{x} \sim N(\mu, \sigma)$ and $\tilde{\theta} \sim N(0,1)$. If $\tilde{x}$ and $\tilde{\theta}$ are jointly normally distributed, then the posterior mean and variance of $\tilde{x}$ given a signal θ are $\mu_\theta = \mu + \rho\sigma\theta$ and $\sigma_\theta^2 = \sigma^2(1-\rho^2)$, respectively. Therefore, the posterior certain equivalent of $\tilde{x}$ is $\hat{x}_\theta = \mu + \rho\sigma\theta - \sigma^2(1-\rho^2)/2R$. This certain equivalent will be greater than v as long as $\theta > (\rho\sigma)^{-1}(v-\mu+\sigma^2(1-\rho^2)/2R)$. Let

$$a = \frac{v-\mu}{\rho\sigma} + \frac{\sigma^2(1-\rho^2)}{2R\rho\sigma} = \frac{1}{\rho\sigma}\left(v - \left(\mu - \frac{\sigma^2(1-\rho^2)}{2R}\right)\right) = \frac{v - E[\hat{x}_\theta]}{\rho\sigma}.$$

The expected utility with imperfect information is then

$$\begin{aligned} u(VWI_\rho) &= \int_{-\infty}^{\infty} u(\max(v,\hat{x}_\theta)) f(\theta)\,d\theta \\ &= \int_{-\infty}^{a} u(v) f(\theta)\,d\theta + \int_{a}^{\infty} u(\hat{x}_\theta) f(\theta)\,d\theta \\ &= -\text{sgn}(R)\int_{-\infty}^{a} e^{-v/R} f(\theta)\,d\theta \\ &\quad -\text{sgn}(R)\int_{a}^{\infty} e^{-(\mu/R+\rho\sigma\theta/R-\sigma^2(1-\rho^2)/(2R^2))} f(\theta)\,d\theta \\ &= -\text{sgn}(R)e^{-v/R}(2\pi)^{-1/2}\int_{-\infty}^{a} e^{-(1/2)\theta^2}\,d\theta \\ &\quad -\text{sgn}(R)e^{-(\mu/R-\sigma^2(1-\rho^2)/(2R^2))} \\ &\quad \cdot (2\pi)^{-1/2}\int_{a}^{\infty} e^{-(1/2)\theta^2-(\rho\sigma/R)\theta}\,d\theta \\ &= -\text{sgn}(R)e^{-v/R}\Phi(a) - \text{sgn}(R) \\ &\quad \cdot e^{-(\mu/R-\sigma^2(1-\rho^2)/(2R^2))} e^{(1/2)(\rho\sigma/R)^2}\Phi\left(-a - \frac{\rho\sigma}{R}\right). \end{aligned}$$

The last integral follows for the same reasons discussed in the previous section. The term in the second exponential can be written as

$$\begin{aligned} &\frac{\mu}{R} - \frac{\sigma^2(1-\rho^2)}{2R^2} - \frac{1}{2}\left(\frac{\rho\sigma}{R}\right)^2 \\ &= \frac{\mu - (1/2)(\sigma^2/R) + (1/2)(\rho^2\sigma^2/R) - (1/2)(\rho^2\sigma^2/R)}{R} \\ &= \frac{\mu - (1/2)(\sigma^2/R)}{R} = \frac{\hat{x}}{R}. \end{aligned}$$

Thus,

$$\begin{aligned} u(VWI_\rho) &= u(v)\Phi(a) + u(\hat{x})\Phi\left(-a - \frac{\rho\sigma}{R}\right) \\ &= u(v)\Phi(a) + u(\hat{x})[1 - \Phi(a + \rho k)]. \end{aligned}$$

Substituting for a, we have

$$\begin{aligned} u(VWI_\rho) &= u(v)\Phi\left(\frac{v - E[\hat{x}_\theta]}{\rho\sigma}\right) \\ &\quad + u(\hat{x})\left[1 - \Phi\left(\frac{v - E[\hat{x}_\theta]}{\rho\sigma} + \rho k\right)\right]. \end{aligned}$$

Because

$$\frac{v - E[\hat{x}_\theta]}{\rho\sigma} = \frac{v-\mu}{\rho\sigma} + \frac{\sigma^2(1-\rho^2)}{2R\rho\sigma} = -\frac{c}{\rho} + \frac{k}{\rho}\frac{(1-\rho^2)}{2} \quad \text{and}$$

$$-\frac{c}{\rho} + \frac{k}{\rho}\frac{1-\rho^2}{2} + \rho k = -\frac{c}{\rho} + \frac{k}{\rho}\frac{1+\rho^2}{2},$$

the expected utility with information can be expressed more compactly as

$$u(VWI_\rho) = u(v)\Phi\left(-\frac{c}{\rho} + \frac{k}{\rho}\frac{1-\rho^2}{2}\right) + u(\hat{x})\Phi\left(\frac{c}{\rho} - \frac{k}{\rho}\frac{1+\rho^2}{2}\right).$$

A5. Value of Imperfect Information
If $\hat{x} \geq v$, then

$$\begin{aligned} &VOI_\rho \\ &= -R\ln\left[e^{-v/R}\Phi\left(-\frac{c}{\rho}+\frac{k}{\rho}\frac{1-\rho^2}{2}\right) + e^{-\hat{x}/R}\Phi\left(\frac{c}{\rho}-\frac{k}{\rho}\frac{1+\rho^2}{2}\right)\right] - \hat{x} \\ &= -R\ln\left[e^{-v/R}\Phi\left(-\frac{c}{\rho}+\frac{k}{\rho}\frac{1-\rho^2}{2}\right)\right. \\ &\qquad \left. + e^{-\hat{x}/R}\Phi\left(\frac{c}{\rho}-\frac{k}{\rho}\frac{1+\rho^2}{2}\right)\right] - R\ln[e^{\hat{x}/R}] \\ &= -R\left[\ln\left[e^{-v/R}\Phi\left(-\frac{c}{\rho}+\frac{k}{\rho}\frac{1-\rho^2}{2}\right)\right.\right. \\ &\qquad \left.\left. + e^{-\hat{x}/R}\Phi\left(\frac{c}{\rho}-\frac{k}{\rho}\frac{1+\rho^2}{2}\right)\right] + \ln[e^{\hat{x}/R}]\right] \\ &= -R\ln\left[e^{\hat{x}/R}e^{-v/R}\Phi\left(-\frac{c}{\rho}+\frac{k}{\rho}\frac{1-\rho^2}{2}\right)\right. \\ &\qquad \left. + e^{\hat{x}/R}e^{-\hat{x}/R}\Phi\left(\frac{c}{\rho}-\frac{k}{\rho}\frac{1+\rho^2}{2}\right)\right] \\ &= -R\ln\left[e^{(\hat{x}-v)/R}\Phi\left(-\frac{c}{\rho}+\frac{k}{\rho}\frac{1-\rho^2}{2}\right) + \Phi\left(\frac{c}{\rho}-\frac{k}{\rho}\frac{1+\rho^2}{2}\right)\right] \\ &= -R\ln\left[e^{ck-k^2/2}\Phi\left(-\frac{c}{\rho}+\frac{k}{\rho}\frac{1-\rho^2}{2}\right) + \Phi\left(\frac{c}{\rho}-\frac{k}{\rho}\frac{1+\rho^2}{2}\right)\right]. \end{aligned}$$

Note: The factor $\text{sgn}(R)$ cancels in the certain equivalent formula.

If $\hat{x} < v$, then

$$\begin{aligned} VOI_\rho &= -R\ln\left[e^{-v/R}\Phi\left(-\frac{c}{\rho}+\frac{k}{\rho}\frac{1-\rho^2}{2}\right)\right. \\ &\qquad \left. + e^{-\hat{x}/R}\Phi\left(\frac{c}{\rho}-\frac{k}{\rho}\frac{1+\rho^2}{2}\right)\right] - v \\ &= -R\ln\left[e^{-v/R}\Phi\left(-\frac{c}{\rho}+\frac{k}{\rho}\frac{1-\rho^2}{2}\right)\right. \\ &\qquad \left. + e^{-\hat{x}/R}\Phi\left(\frac{c}{\rho}-\frac{k}{\rho}\frac{1+\rho^2}{2}\right)\right] - R\ln[e^{v/R}] \end{aligned}$$

$$= -R\ln\left[e^{v/R}e^{-v/R}\Phi\left(-\frac{c}{\rho}+\frac{k}{\rho}\frac{1-\rho^2}{2}\right) + e^{v/R}e^{-\hat{x}/R}\Phi\left(\frac{c}{\rho}-\frac{k}{\rho}\frac{1+\rho^2}{2}\right)\right]$$

$$= -R\ln\left[\Phi\left(-\frac{c}{\rho}+\frac{k}{\rho}\frac{1-\rho^2}{2}\right) + e^{-ck+k^2/2}\Phi\left(\frac{c}{\rho}-\frac{k}{\rho}\frac{1+\rho^2}{2}\right)\right].$$

A6. TALL-N Solution

Let $\gamma = R^{-1}$. Assuming $\hat{x} \geq v$ and taking the derivative of $1+u(VOI_\rho)$ with respect to γ, we have

$$\frac{d}{d\gamma}(1+u(VOI_\rho))$$
$$=-\text{sgn}(\gamma)\frac{d}{d\gamma}\left[e^{-\gamma v+\gamma\mu-\gamma^2\sigma^2/2}\Phi\left(-\frac{c}{\rho}+\frac{\gamma\sigma}{\rho}\frac{1-\rho^2}{2}\right) +\Phi\left(\frac{c}{\rho}-\frac{\gamma\sigma}{\rho}\frac{1+\rho^2}{2}\right)\right]$$
$$=-\text{sgn}(\gamma)\left[(-v+\mu-\gamma\sigma^2)e^{-\gamma v+\gamma\mu-\gamma^2\sigma^2/2}\Phi\left(-\frac{c}{\rho}+\frac{\gamma\sigma}{\rho}\frac{1-\rho^2}{2}\right) +e^{-\gamma(v-\mu+\gamma\sigma^2/2)}\phi\left(-\frac{c}{\rho}+\frac{\gamma\sigma}{\rho}\frac{1-\rho^2}{2}\right)\frac{\sigma}{\rho}\frac{1-\rho^2}{2} -\phi\left(\frac{c}{\rho}-\frac{\gamma\sigma}{\rho}\frac{1+\rho^2}{2}\right)\frac{\sigma}{\rho}\frac{1+\rho^2}{2}\right].$$

As $\gamma \to 0$, this derivative approaches $\rho\sigma\phi(\rho^{-1}c) - (\mu - v)\cdot \Phi(-\rho^{-1}c)$. Since $d(1-e^{-\gamma})/d\gamma = e^{-\gamma}$ approaches 1 as $\gamma \to 0$,

$$\lim_{\gamma\to 0} VOI_\rho = EVI_\rho = \rho\sigma\phi(\rho^{-1}c) - (\mu - v)\Phi(-\rho^{-1}c)$$
$$= \rho\sigma[\phi(\rho^{-1}c) - \rho^{-1}c\Phi(-\rho^{-1}c)].$$

The case $\hat{x} < v$ is solved similarly.

A7. Expected Value of Imperfect Information Under Lognormal Distribution

Suppose $\tilde{x}$ is lognormally distributed such that $\ln\tilde{x} \sim N(\mu', \sigma')$. The mean of $\tilde{x}$ is $\mu = e^{\mu'+(\sigma')^2/2}$ and the variance is $\sigma^2 = (e^{(\sigma')^2}-1)e^{2\mu'+(\sigma')^2}$. Following the same procedures outlined above, one can show that the expected value of imperfect information is

$$EVI_\rho = \begin{cases} v\Phi\left(-\frac{c}{\rho}-\frac{\sigma'}{\rho}\frac{1-\rho^2}{2}\right)-\mu\Phi\left(-\frac{c}{\rho}-\frac{\sigma'}{\rho}\frac{1+\rho^2}{2}\right) & \mu \geq v \\ -v\Phi\left(\frac{c}{\rho}+\frac{\sigma'}{\rho}\frac{1-\rho^2}{2}\right)+\mu\Phi\left(\frac{c}{\rho}+\frac{\sigma'}{\rho}\frac{1+\rho^2}{2}\right) & \mu < v, \end{cases}$$

where $c = (\mu' - \ln v)/\sigma'$. We see that in the lognormal case EVI_ρ/EVI_1 is no longer a function solely of c and ρ.

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Correction

On page 119 of this article the text in the last paragraph was corrected to read:

Without loss of generality, we assume that the information signal is unit normal and therefore the conditional mean of $\tilde{x}$ given θ is $E[\tilde{x} \mid \theta] = \mu + \rho\sigma\theta$, which will exceed v as long as $\theta > (v-\mu)/(\rho\sigma) = -\rho^{-1}c$.